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Stability of Isotropic Incompressible Turbulence Against Weak Mean Flow Perturbations

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The stability of incompressible turbulent fluids with respect to weak mean flow perturbations is discussed. It is shown that for a statistically homogeneous, isotropic, and stationary model such perturbations will decay. This is in marked contrast to the compressible case.

KEY WORDS: Turbulence; self-organization; helicity; incompressible fluids.

When describing the phenomenon of turbulence, much emphasis is often placed on ensembles of flows which are incompressible and, in a statistical sense, homogeneous, isotropic, and stationary. Whether or not such turbulence exists in real flows is a moot point, but certainly this form is mathematically more tractable. It is also often stated that turbulence resulting from the flow of a turbulent fluid past a grid approximates to being homogeneous and isotropic.(1) In a wide variety of experiments, however, large-scale coherent vortex structures are observed.(2) This presents an intriguing challenge to theorists: what is a possible mechanism which would enable ordered structures to be formed from the seemingly chaotic motion of a turbulent fluid? Moreover, does the form of turbulence described above represent an adequate model for describing the formation and evolution of such structures? To pose the question in a more physical way: consider an ensemble of turbulent flows on which, at some time t_0, in every realization we impose the same large-scale velocity pattern $\langle \mathbf{v}(\mathbf{x}, t_0) \rangle$; how will this ensemble evolve? Will the perturbation grow with

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time and will concomitant vortex growth ensue? If so, upon what physical parameters of the unperturbed turbulent flows will this depend?

The stability of a variety of small-scale flows, such as ABC and Kolmogorov flows, subjected to large-scale perturbations has recently received close attention in a number of papers. Shtilman and Sivashinsky[3,4] have considered simple periodic flows and have used asymptotic expansions to show that, for large Reynolds numbers, such flows could be unstable to long-wavelength perturbations, a result also obtained by Galloway and Frisch[5] in numerical simulations. Bayly[6] has suggested, using Floquet theory, that quasi-two-dimensional flows in which the streamlines all form closed curves could be subject to broad-band instabilities. All these predicted instabilities may have their origin in the fundamental anisotropy of the basic flows considered, as research by Sivashinsky and Yakhot,[7] Bayly and Yakhot,[8] Pouquet *et al.*,[9] Kraichnan,[10] and Yakhot and Pelz[11] suggests.

Other authors have discussed the effect that helicity and helicity density may have upon the stability of turbulent fluid systems. Moffatt,[12] using the equivalence between the MHD equations and the Navier–Stokes equation, showed that Beltrami flows are unstable against perturbations whose helicity is of the same sign as that of the basic flow. Pouquet *et al.*[9] have used renormalization group (RG) methods to show that helical forcing can introduce an additional term into the Navier–Stokes equation; they show, however, that this term is negligible at large scales. An interesting and informative article by Moiseev *et al.*[13],[4] combined the two physical aspects mentioned above: they looked at an ensemble of turbulent, compressible fluids which were statistically homogeneous, isotropic, and stationary and possessed a nonzero mean helicity density; they then subjected the ensemble to large-scale velocity and density perturbations. It is instructive to examine their final equations closely. These equations are

$$\frac{\partial\langle\mathbf{v}\rangle}{\partial t}=\frac{g(0)}{2}\nabla\times\langle\mathbf{v}\rangle+\left[\nu+\frac{C(0)}{2}\right]\nabla^2\langle\mathbf{v}\rangle-\frac{c_0^2}{\rho_0}\nabla\langle\rho\rangle \tag{1}$$

for the velocity, where here and henceforth $\langle\cdots\rangle$ indicate ensemble averages and where c_0 is the sound speed, which in the model of ref. 13 is a constant, and

$$\frac{\partial\langle\rho\rangle}{\partial t}+\nabla\cdot(\langle\rho\rangle\langle\mathbf{v}\rangle)+D(0)\nabla\cdot\langle\mathbf{v}\rangle=C(0)\,\nabla^2\langle\rho\rangle \tag{2}$$

[4] Ref. 13 contains a number of numerical errors.

for the density, where the unperturbed turbulent velocity and density fields $\mathbf{v}^{\mathrm{t}}$ and ρ^{t}, respectively, are correlated as follows (here and henceforth summation over repeated indices is implied):

$$\begin{aligned}\langle v_i^{\mathrm{t}}(\mathbf{x}, t)\, v_j^{\mathrm{t}}(\mathbf{x}', t') &= C_{ij}(\mathbf{x}-\mathbf{x}')\,\phi(t-t') \\ &= [C(r)\,\delta_{ij} + B(r)\,r_i r_j + g(r)\,\varepsilon_{ijk} r_k]\,\phi(\tau) \\ \langle v_i^{\mathrm{t}}(\mathbf{x}, t)\,\rho^{\mathrm{t}}(\mathbf{x}', t')\rangle &= D_i(\mathbf{x}-\mathbf{x}')\,\phi(t-t') \\ &= D(r)\,r_i\phi(\tau), \qquad \mathbf{r}=\mathbf{x}-\mathbf{x}', \quad \tau = t-t'\end{aligned} \tag{3}$$

where ε_{ijk} is the totally antisymmetric third-rank unit tensor. As the ensemble considered is assumed to be statistically homogeneous, isotropic, and stationary, these are the most general forms that two-point correlators can take.(13) If $g(r) \neq 0$ for some r, then the ensemble will lack reflectional symmetry. Moreover, it follows from Eq. (3) that the mean helicity density $\langle(\mathbf{v}^{\mathrm{t}}\cdot\nabla\times\mathbf{v}^{\mathrm{t}})\rangle$ equals $6g(0)\,\phi(0)$. Moiseev *et al.*(13) chose $g(0)\neq 0$, so that there is a mean helicity density, and thus a preferred "handedness" for the system. They also assume that there exists a time τ_c, the correlation time of the turbulent velocity field, such that $\phi(\tau)=0$ when $\tau \geqslant \tau_c$. In their equations, the bulk viscosity term $\zeta\nabla(\nabla\cdot\langle\mathbf{v}\rangle)$ was neglected. For consistency, they should have neglected the terms which are quadratic in the averaged quantities, since in their derivation these have already been assumed to be small. Furthermore, the condition $\langle(\mathbf{v}^{\mathrm{t}}\cdot\nabla)\,\rho^{\mathrm{t}}\rangle = 0$ requires(1) that $D(0)=0$. The second of Eqs. (3) thus becomes

$$\partial\langle\rho\rangle/\partial t = C(0)\,\nabla^2\langle\rho\rangle \tag{4}$$

Since it follows from Eq. (3) that $C(0)$ is positive definite, the density perturbation decays, but Fourier transforming Eq. (1) and assuming the usual plane-wave dependence of the perturbations, one finds that vortex growth is possible in the following wavenumber range:

$$0 \leqslant k \leqslant \frac{g(0)}{2\nu + C(0)} \tag{5}$$

This is a purely kinematic effect; an inviscid fluid could be considered in which the pressure is constant: each fluid particle would then move along a streamline with constant velocity, but the instability would still occur. The equation derived by Moiseev *et al.*(13) is, however, not a feasible form for perturbations to an incompressible fluid, despite the lack of dependence on the pressure term—and hence on the equation of state. To see this, it should be remembered that angular momentum is conserved in every realization of the flow, and thus it should also be conserved after ensemble averaging. This is not the case for an incompressible fluid if Eq. (1) is valid, even if nonlinear terms are included (see Appendix). It is therefore interesting to investigate the question of vortex formation in homogeneous, isotropic, incompressible turbulence in close detail, employing the simple technique developed in ref. 13, and to trace, if possible, where the derivation of Eq. (1) breaks down for an incompressible fluid.

Consider an ensemble of incompressible fluids, the nth realization of which has a velocity field $\mathbf{v}_n^{\mathrm{t}}$. For the sake of simplicity, we shall drop in what follows the index n. This velocity field is a solution of the Navier–Stokes equation, so that

$$\partial_t v_i^{\mathrm{t}} + v_j^{\mathrm{t}}\, \partial_j v_i^{\mathrm{t}} = \nu\, \partial^2 v_i^{\mathrm{t}} - \partial_i p^{\mathrm{t}} + F_i \tag{6}$$

where we have put

$$\partial_t \equiv \partial/\partial t, \qquad \partial_i \equiv \partial/\partial x_i, \qquad \partial^2 \equiv \nabla^2 \tag{7}$$

In Eq. (6), $\mathbf{v}^{\mathrm{t}}$, $\mathbf{x}$, and t are, respectively, the local turbulent fluid velocity, the position, and the time; the density has been put equal to unity. Equation (6) gives, if we take its divergence, an equation for the pressure p^{t} which closes the set of equations. This equation is

$$\partial^2 p^{\mathrm{t}} = (\partial_i v_j^{\mathrm{t}})(\partial_j v_i^{\mathrm{t}}) \tag{8}$$

We assume that the statistics for the turbulent fields is still given by the first of Eq. (3), but we shall not assume that $g(0)$ is necessarily nonzero. We now impose at some time $t = t_0$ the same perturbation $\langle \mathbf{v}(\mathbf{x}, t_0) \rangle$ on every realization, so that at any subsequent time the velocity and pressure fields $\mathbf{v}$ and p can be represented by

$$\mathbf{v} = \langle \mathbf{v} \rangle + \mathbf{v}^{\mathrm{t}} + \mathbf{v}^{1}; \qquad p = \langle p \rangle + p^{\mathrm{t}} + p^{1} \tag{9}$$

in which one can interpret $\mathbf{v}^1$ and p^1 as interaction terms. These have been introduced since it is not, in general, possible to decompose the Navier–Stokes equation into a large-scale part and a fluctuating remainder, of which the latter is then further assumed to obey the Navier–Stokes equation. The two interaction terms are initially zero and are of zero mean. They are thus well defined for all subsequent times. The total velocity field obeys the Navier–Stokes equation as well as Eqs. (9), so that an equation for the evolution of the perturbation can be obtained by inserting Eqs. (9) into the Navier–Stokes equation and then performing the ensemble average. As we are free to consider small perturbations, so that the mean and interaction fields are much smaller in magnitude than the basic flow—at least in the initial stages—the resulting expression may be linearized and written as

$$\partial_t \langle v_i \rangle + \partial_j \langle v_i^t v_j^1 + v_j^t v_i^1 \rangle = \nu\, \partial^2 \langle v_i \rangle - \partial_i \langle p \rangle \tag{10}$$

and is of a form analogous to the Reynolds stress equation (see, e.g., ref. 14). The pressure equation can be found simply by taking the divergence of Eq. (10) and gives

$$\partial^2 \langle p \rangle = 2 \partial_i \partial_j \langle v_i^t v_j^1 \rangle \tag{11}$$

Equations (10) and (11) will form a closed set once we have calculated the correlators

$$Q_{ij}(\mathbf{x}, t) = \langle v_i^t(\mathbf{x}, t)\, v_j^1(\mathbf{x}, t) \rangle \tag{12}$$

The closure problem has arisen, as usual, from the averaging procedure; to progress further, we must make additional assumptions about the nature of the turbulence. We shall assume that $\mathbf{v}^t$ obeys Gaussian statistics. The Gaussian approximation is arguably not particularly good for describing real flows, but it does hold well for large-scale components of the velocity field (see, e.g., ref. 15). Moreover, this statistical treatment should be at least not worse than the deterministic models used before in investigating the problem of mean flow stability. As $\mathbf{v}^1$ is a functional of $\mathbf{v}^t$, we then find from the Novikov–Furutsu formula[16,17] and Eq. (3) that

$$Q_{ij}(\mathbf{x}, t) = \int d^3\mathbf{x}'\, dt'\, C_{ik}(\mathbf{x} - \mathbf{x}')\, \phi(t - t') \left\langle \frac{\delta v_j^1(\mathbf{x}, t)}{\delta v_k^t(\mathbf{x}, t)} \right\rangle \tag{13}$$

In order to evaluate the functional derivatives in Eq. (13), we must find an equation for the interaction fields. If we substitute Eqs. (9) into the Navier–Stokes equation and use Eq. (10), we obtain after some simplifications

$$\partial_t \mathbf{v}^1 + (\langle \mathbf{v} \rangle \cdot \nabla)\, \mathbf{v}^t + (\mathbf{v}^t \cdot \nabla) \langle \mathbf{v} \rangle = -\nabla p' + \mathbf{Z}\{\mathbf{v}^t\} \tag{14}$$

where $\mathbf{Z}\{\mathbf{v}^t\}$ is a known functional of $\mathbf{v}^t$. A formal solution of Eq. (14) can be obtained by integrating it over the time. We write the result in the following form:

$$\begin{aligned} v_j^1(\mathbf{x}, t) = \int_0^t dt' \Bigg[& -v_q^t(\mathbf{x}, t') \frac{\partial \langle v_j(\mathbf{x}, t') \rangle}{\partial x_q} - \langle v_q(\mathbf{x}, t') \rangle \frac{\partial v_j^t(\mathbf{x}, t')}{\partial x_q} \\ & - \frac{\partial p'(\mathbf{x}, t')}{\partial x_j} + Z_j \Bigg] \end{aligned} \tag{15}$$

Obtaining the functional derivative is then straightforward:

$$\left\langle \frac{\delta v_j^1(\mathbf{x}, t)}{\delta v_k^t(\mathbf{x}', t')} \right\rangle = -\Theta(t-t') \left\{ \delta(\mathbf{x}-\mathbf{x}') \frac{\partial \langle v_j(\mathbf{x}, t') \rangle}{\partial x_q} + \langle v_q(\mathbf{x}, t') \rangle \frac{\partial}{\partial x_q} [\delta_{kj} \delta(\mathbf{x}-\mathbf{x}')] \right\} - \int_0^t dt'' \left[\frac{\partial}{\partial x_j} \left\langle \frac{\delta p^1(\mathbf{x}, t'')}{\delta v_k^t(\mathbf{x}', t')} \right\rangle + \left\langle \frac{\delta Z_j}{\delta v_k} \right\rangle \right] \tag{16}$$

where $\Theta(t-t')$ is the Heaviside step funtion and where

$$\int_0^t dt'' \left\langle \frac{\delta Z_j}{\delta v_k} \right\rangle = \int_0^t dt'' \left[\nu\, \partial^2 \left\langle \frac{\delta v_j^1(\mathbf{x}, t'')}{\delta v_k^t(\mathbf{x}', t')} \right\rangle + \left\langle \frac{\delta v_q^1(\mathbf{x}, t)}{\delta v_k^t(\mathbf{x}', t')} \frac{\partial v_i^t(\mathbf{x}, t'')}{\partial x_q} \right\rangle - \left\langle v_q^t(\mathbf{x}, t') \frac{\partial}{\partial x_q} \left(\frac{\delta v_j^1(\mathbf{x}, t'')}{\delta v_k^t(\mathbf{x}', t')} \right) \right\rangle \right] \tag{17}$$

It can be shown that the right-hand side of Eq. (17) can be neglected, provided

$$\frac{|\mathbf{v}^t| \min(\lambda, L)}{\nu} \gg 1 \qquad \text{and} \qquad \frac{\min(\lambda, L)}{|\mathbf{v}^t|\, \tau_c} \gg 1 \tag{18}$$

in which λ and L are the length scales of the background turbulence and of the perturbations, respectively. We note that the conditions (18) do not necessarily restrict us to the case of large-scale perturbations when $\lambda/L \ll 1$. The first of these inequalities means that the Reynolds numbers for the unperturbed and for the average velocities must be large, so that we are dealing with truly turbulent flows. The second inequality is a condition that the time scales on which these velocity fields change should be much longer than the correlation time of the basic flow.

The functional derivatives of the interaction pressure with respect to the turbulent velocity field can be evaluated in a similar manner and we get, to the same approximation as before,

$$\left\langle \frac{\delta p^1(\mathbf{x}, t'')}{\delta v_k^t(\mathbf{x}', t')} \right\rangle = 2 \frac{\partial G(\mathbf{x}-\mathbf{x}')}{\partial x_p'} \frac{\partial \langle v_p(\mathbf{x}', t'') \rangle}{\partial x_k'} \delta(t'-t'') \tag{19}$$

Finally, this equation is integrated over t'' to give

$$\int_0^t dt'' \left\langle \frac{\delta p^1(\mathbf{x}, t'')}{\delta v_k^t(\mathbf{x}', t')} \right\rangle = 2\Theta(t-t') \frac{\partial \langle v_p(\mathbf{x}', t') \rangle}{\partial x_k'} \frac{\partial G(\mathbf{x}-\mathbf{x}')}{\partial x_p'} \tag{20}$$

In Eqs. (19) and (20), $G(\mathbf{x}-\mathbf{x}')$ ($=-1/4\pi\,|\mathbf{x}-\mathbf{x}'|$) is the Green function for the Laplace equation. The response of the interaction velocity to a change in the turbulent velocity field is thus, when averaged,

$$\left\langle \frac{\delta v_j^{\mathrm{I}}(\mathbf{x},t)}{\delta v_k^{\mathrm{t}}(\mathbf{x}',t')} \right\rangle = -\Theta(t-t')\left\{\delta(\mathbf{x}-\mathbf{x}')\frac{\partial\langle v_j(\mathbf{x},t')\rangle}{\partial x_k} + \langle v_q(\mathbf{x},t')\rangle\left[\frac{\partial}{\partial x_q}\delta_{kj}\delta(\mathbf{x}-\mathbf{x}')\right] + 2\frac{\partial G(\mathbf{x}-\mathbf{x}')}{\partial x_p'}\frac{\partial\langle v_p(\mathbf{x}',t')\rangle}{\partial x_k'}\right\} \tag{21}$$

As the step function is zero for $t<t'$, causality is not violated. We can now insert the functional derivative (21) into Eq. (13) and perform the integrations for the first two terms to obtain

$$Q_{ij}(\mathbf{x},t) = -\int dt'\,\Theta(t-t')\,\phi(t-t')\,C_{ik}(0)\frac{\partial\langle v_j(\mathbf{x},t')\rangle}{\partial x_k} - 2\int dt'\,d^3\mathbf{x}'\,\Theta(t-t')\,\phi(t-t')\,C_{ik}(\mathbf{x}-\mathbf{x}') \times \frac{\partial}{\partial x_j}\left[\frac{\partial G(\mathbf{x}-\mathbf{x}')}{\partial x_p'}\frac{\partial\langle v_p(\mathbf{x}',t')\rangle}{\partial x_k'}\right] \tag{22}$$

Using Eq. (3), we can now perform the first integral in Eq. (22) and if we additionally assume that $\phi(\tau)$ is a delta function, we get

$$Q_{ij}(\mathbf{x},t) = -\frac{1}{2}C(0)\frac{\partial\langle v_j\rangle}{\partial x_i} - \int d^3\mathbf{x}'\,\frac{\partial\langle v_p(\mathbf{x}',t)\rangle}{\partial x_k'}\left(C_{ik}\frac{\partial^2}{\partial x_j\,\partial x_p'} + C_{jk}\frac{\partial^2}{\partial x_i\,\partial x_p'}\right)G \tag{23}$$

The evolution for the perturbation can thus be found by inserting Eq. (23) into Eq. (10). This gives

$$\partial_t\langle v_i\rangle = \left[\nu + \frac{C(0)}{2}\right]\nabla^2\langle v_i\rangle - \partial_i\langle p\rangle + \partial_j\int d^3\mathbf{x}'\left(C_{ik}\frac{\partial^2}{\partial x_j\,\partial x_p'} + C_{jk}\frac{\partial^2}{\partial x_i\,\partial x_p'}\right)G \tag{24}$$

While this may look complicated, it should be borne in mind that the only thing we are interested in at the moment is a dispersion relation for the growth of the perturbation, that is, a relationship between the growth rate γ and the wavevector $\mathbf{k}$. To find this relation, we notice that both the Green function for the Laplace equation and the two-point velocity correlator depend solely on the separation between two points, that is, on $\mathbf{r} = \mathbf{x} - \mathbf{x}'$. Moreover, Eq. (24) is linear in $\langle \mathbf{v} \rangle$. Hence, Eq. (24) can be regarded as an integral equation for $\langle \mathbf{v} \rangle$ which possesses a displacement kernel. The standard procedure to deal with such equations (see, e.g., ref. 18) is to Fourier transform. This gives

$$\Gamma(\mathbf{k}, \omega)\, u_i(\mathbf{k}, \omega) = ik_j F_{ij}(\mathbf{k}, \omega) + ik_i\, p(\mathbf{k}, \omega) \tag{25}$$

where we have written

$$\Gamma(\mathbf{k}, \omega) = -i\omega + \left[\nu + \frac{C(0)}{2} \right] k^2 \tag{26a}$$

$$p(\mathbf{k}, \omega) = \mathrm{F.T.}[\langle p(\mathbf{x}, t) \rangle] \tag{26b}$$

$$u_i(\mathbf{k}, \omega) = \mathrm{F.T.}[\langle v_i(\mathbf{x}, t) \rangle] \tag{26c}$$

$$F_{ij}(\mathbf{k}, \omega) = \mathrm{F.T.} \left\{ \int d^3\mathbf{x}' \left[\left(C_{ik} \frac{\partial^2}{\partial x_j\, \partial x'_p} + C_{jk} \frac{\partial^2}{\partial x_i\, \partial x'_p} \right) G \right] \times \frac{\partial \langle v_p(\mathbf{x}', t') \rangle}{\partial x'_k} \right\} \tag{26d}$$

As the term containing the integral is a convolution, Fourier transformed it produces a product

$$F_{ij}(\mathbf{k}, \omega) = ik_q u_p \times \mathrm{F.T.} \left\{ \left[-C_{iq}(r) \frac{\partial^2}{\partial r_j\, \partial r_p} - C_{jq}(r) \frac{\partial^2}{\partial r_i\, \partial r_p} \right] G(r) \right\} \tag{27}$$

To evaluate the Fourier transforms which remain in Eq. (27), we Fourier transform the quantities $C_{ij}(r)$ and $G(r)$ separately and, noting that we will obtain a convolution, obtain the coefficient of u_p as an integral over Fourier space. We have

$$C_{iq}(r) \frac{\partial^2 G(r)}{\partial r_j\, \partial r_p} = \int d^3\mathbf{k}'\, d^3\mathbf{k}''\, \Phi_{iq}(\mathbf{k}'') \frac{k'_j k'_p}{k'^2} \exp[i(\mathbf{k}' + \mathbf{k}'') \cdot \mathbf{r}] \tag{28}$$

so that

$$\mathrm{F.T.} \left[C_{iq}(r) \frac{\partial^2 G}{\partial r_j\, \partial r_p} \right] = \int d^3\mathbf{k}'\, \frac{\Phi_{iq}(\mathbf{k}')(\mathbf{k} - \mathbf{k}')_j\, (\mathbf{k} - \mathbf{k}')_p}{|\mathbf{k} - \mathbf{k}'|^2} \tag{29}$$

where we have introduced the energy spectrum tensor[1] through the equation

$$\Phi_{iq}(\mathbf{k}) = \text{F.T.}[C_{iq}(\mathbf{r})] \tag{30}$$

and used the fact that $\text{F.T.}[G(r)] = 1/k^2$. If we also use the incompressibility condition $(\mathbf{k} \cdot \mathbf{u}) = 0$, we get finally from Eq. (25)

$$\Gamma u_i = ik_i p - k_j k_q u_p \left\{ d^3\mathbf{k}' \frac{(k_j' k_p' - k_j k_p)\, \Phi_{iq}(\mathbf{k}') + (k_i' k_p' - k_i k_p)\, \Phi_{jq}(\mathbf{k}')}{|\mathbf{k} - \mathbf{k}'|^2} \right\} \tag{31}$$

It is known, however, that for homogeneous, isotropic turbulence the most general form for the energy spectrum tensor is[19]

$$\Phi_{ik}(\mathbf{q}) = \frac{E(q)}{4\pi q^4}(q^2 \delta_{ik} - q_i q_k) + \frac{iF(q)}{8\pi q^4} \varepsilon_{iks} q_s \tag{32}$$

where $E(q)$ and $F(q)$ are, respectively, the energy and helicity spectral densities which satisfy the relations

$$\langle v^2/2 \rangle = \int dq\, E(q), \qquad \langle \mathbf{v} \cdot \boldsymbol{\omega} \rangle = \int dq\, F(q) \tag{33}$$

where $\boldsymbol{\omega} = \nabla \times \mathbf{v}$. One of the consequences of Eqs. (33), which follows from the Schwartz inequality, is that for any wavenumber q[19]

$$F(q) \leqslant 2qE(q) \tag{34}$$

a relation which will be important in what follows. It is instructive to write Eq. (31) in the form

$$\Gamma u_i = -T_{ip} u_p + ik_i p \tag{35}$$

and then to observe that, as $T_{ip}(k)$ is a second-rank tensor, from its definition it should be possible to express it in the form

$$T_{ip}(k) = K(k)\, \delta_{ip} + L(k)\, k_i k_p + M(k)\, \varepsilon_{ips} k_s \tag{36}$$

When this is inserted into Eq. (35) we get, after again using the incompressibility condition,

$$\Gamma \mathbf{u} = K(k)\mathbf{u} - M(k)\mathbf{k} \times \mathbf{u} + i\mathbf{k}p \tag{37}$$

If we take the scalar product of this equation with $\mathbf{k}$, we obtain

$$k^2 p = 0 \qquad \text{or} \qquad \nabla^2 p = 0 \tag{38}$$

Since we require that the pressure gradient tends to zero at infinity and also remains finite at the origin, the solution of Eq. (38) is simply $p = \text{const.}$ Thus, the equation for the perturbation can be written in the form

$$(\Gamma - K)\mathbf{u} = -M\mathbf{k} \times \mathbf{u} \tag{39}$$

The $K(k)$ term can be considered as a turbulent, or eddy, viscosity, while the $M(k)$ term represents vortex generation, since in Fourier space

$$\boldsymbol{\omega}(\mathbf{k}, t) = i\mathbf{k} \times \mathbf{u}(\mathbf{k}, t) \tag{40}$$

Together with the incompressibility condition, Eq. (39) presents us with three linear homogeneous equations for the three velocity components and, in order that these three equations be compatible, we get the dispersion relation

$$\Gamma(\mathbf{k}, \omega) = K(k) \pm ikM(k) \tag{41}$$

This relation replaces Eq. (33) of ref. 13, just as Eq. (39) replaced Eq. (32) of that paper. We must now evaluate $K(k)$ and $M(k)$. To do this, we take our 3-axis along the wavevector $\mathbf{k}$. If we then turn to Eq. (36), we find

$$T_{11} = K(k), \qquad T_{12} = kM(k) \tag{42}$$

or, explicitly,

$$T_{11} = \int d^3\mathbf{q}\, \frac{(k^2 q_3 - k^3)\, q_1 \Phi_{13}(\mathbf{q}) + k^2 q_1^2 \Phi_{33}(\mathbf{q})}{|\mathbf{k} - \mathbf{q}|^2} \tag{43}$$

$$T_{12} = \int d^3\mathbf{q}\, \frac{(k^2 q_2 q_3 - k^3 q_2)\, \Phi_{13}(\mathbf{q}) + k^2 q_1 q_2 \Phi_{33}(\mathbf{q})}{|\mathbf{k} - \mathbf{q}|^2} \tag{44}$$

T_{11} and T_{12} can be (partially) evaluated by introducing spherical coordinates, as one can then integrate analytically over the angle variables. The result is

$$T_{11} = \int dq\, \frac{qE(q)}{4}\, k^2(2qI_4 - kI_3 - 3qI_2 + kI_1 + qI_0) \tag{45}$$

$$T_{12} = i \int dq\, \frac{F(q)}{8}\, k^2(-qI_3 + kI_2 + qI_1 - kI_0) \tag{46}$$

where the integrals I_n are defined by

$$I_n = \int_{-1}^{+1} \frac{c^n\, dc}{k^2 + q^2 - 2kqc} \tag{47}$$

with $c = \cos\theta$, where θ is the angle between $\mathbf{k}$ and $\mathbf{q}$. Such integrals are well documented and recursion relations are readily found[20]:

$$I_n = \frac{k^2+q^2}{2kq} I_{n-1} - \frac{1-(-1)^n}{2kqn} \tag{48}$$

while I_0 is given by the expression

$$I_0 = \frac{1}{kq}\left|\frac{k+q}{k-q}\right| \tag{49}$$

It then follows that

$$T_{11} = \int dq\, E(q)\, pI(p), \qquad T_{12} = \int dq \frac{F(q)}{2} I(p) \tag{50}$$

where $p = q/k$, while $I(p)$ is given by the equation

$$I(p) = \frac{1}{32p^2}\left[(p^2-1)^3 \ln\left|\frac{1+p}{1-p}\right| - 2p(p^2-3)\left(p^2+\frac{1}{3}\right)\right] \tag{51}$$

To determine the stability characteristics of the perturbation, we write down the growth rate γ:

$$\gamma = -k^2\left(v + \frac{\tau_c}{3}\int dq\left\{E(q) + J(p)\left[pE(q) \pm \frac{F(q)}{4}\right]\right\}\right) \tag{52}$$

where $J(p) = 3I(p)$. There will be an instability, resulting in a growth of the perturbation and a production of large-scale vortices, if the right-hand side of Eq. (52) becomes positive for some value of k. We recall, however, that it follows from Eq. (34) that the largest value $\gamma_{\max}$ will occur when the equality sign holds in Eq. (34), so that

$$\gamma_{\max} = -k^2\left\{v + \frac{\tau_c}{3}\int dq\, E(q)[1 + J(p)(p-1)]\right\} \tag{53}$$

Therefore, an instability can occur only if for some p

$$S(p) \equiv 1 + J(p)(p-1) < 0 \tag{54}$$

since we know that $E(q)$ is positive definite. However, we see from Fig. 1 that $S(p)$ is always greater than zero.

We have thus shown that, in a wide range of scales, weak, mean-motion perturbations to an *incompressible*, homogeneous, *isotropic*, and statistically stationary ensemble of fluids will decay; such an ensemble is therefore stable against this class of perturbations. This result agrees with

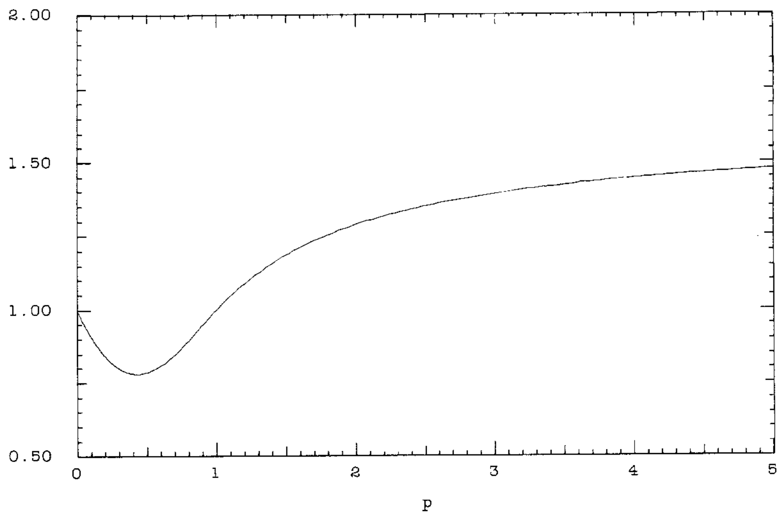

Fig. 1. $S(p)$ as a function of p.

the work by Bayly and Yakhot,[8] whose field-theoretic description of deterministic, small-scale, strong Beltrami flows indicated that the effect of small-scale motion is to "renormalize" the molecular viscosity which, as in our calculations, is still positive for isotropic motion. It also agrees with their result that this effective viscosity is independent of the molecular viscosity at large Reynolds numbers. Furthermore, the results confirm the RG treatment given in ref. 9, which shows that the effect of helicity is to oppose the eddy viscosity term, and thus to inhibit energy transfer to small scales, a result also obtained by Kraichnan.[10] Encouraged by the wide variety of different methods employed by other researchers which yield essentially all the same conclusions, we tentatively suggest that for incompressible fluids large-scale vortex formation requires anisotropy in the unperturbed flow. To investigate this claim further, we hope elsewhere to discuss the calculations presented here in greater detail and also to treat the two-dimensional case for which the fundamental anisotropy (which at the least corresponds to axisymmetry) is simpler to deal with analytically.

APPENDIX

Here we show that Eq. (1) cannot be valid for incompressible flows. To begin with, we write the Navier–Stokes equation in the form

$$\frac{\partial v_i}{\partial t} = \frac{\partial}{\partial x_j}\left[-v_i v_j - p\delta_{ij} + \nu\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)\right] + F_i \equiv \frac{\partial}{\partial x_j}\sigma_{ij} + F_i \tag{A.1}$$

where σ_{ij} is a symmetric tensor, so that $\sigma_{ij}=\sigma_{ji}$. Accordingly, the vorticity equation can be written as

$$\frac{\partial \omega_i}{\partial t}=\varepsilon_{ijk}\frac{\partial^2 \sigma_{kp}}{\partial x_j\,\partial x_p}+\varepsilon_{ijk}\frac{\partial F_k}{\partial x_j} \tag{A.2}$$

It is possible to show that the mean angular momentum, defined by

$$\langle \mathbf{L}\rangle=\rho\int \mathbf{x}\times\langle \mathbf{v}\rangle\, d^3\mathbf{r} \tag{A.3}$$

is conserved if the velocity field obeys the Navier–Stokes equation, but not if it obeys the incompressible version of Eq. (1). However, there are some problems concerning the finiteness of the integral in Eq. (A.3) when the fluid volume is unbounded,[21] so we chose to consider the moment of the impulse required to generate the fluid motion from rest. This is

$$\mathbf{M}=\frac{1}{3}\rho\int \mathbf{x}\times(\mathbf{x}\times\boldsymbol{\omega})\, d^3\mathbf{r} \tag{A.4}$$

and hence

$$\frac{\partial \mathbf{M}}{\partial t}=\frac{1}{3}\rho\int \mathbf{x}\times\left(\mathbf{x}\times\frac{\partial \boldsymbol{\omega}}{\partial t}\right) d^3\mathbf{r} \tag{A.5}$$

Inserting expression (A.2) into (A.5) and using Gauss' divergence theorem, we obtain for the mean impulse

$$\frac{\partial\langle M_i\rangle}{\partial t}=\frac{2}{3}\rho\int \varepsilon_{ijk}\langle\sigma_{jk}\rangle\, d^3\mathbf{r}=0 \tag{A.6}$$

as σ_{ij} is symmetric, ε_{ijk} totally antisymmetric, and $\langle F_i\rangle$ equal to zero. However, if Eq. (1) were valid, we would have

$$\frac{\partial\langle v_i\rangle}{\partial t}=\frac{g(0)}{2}\langle\omega_i\rangle+\left[\nu+\frac{C(0)}{2}\right]\frac{\partial}{\partial x_j}\left(\frac{\partial\langle v_i\rangle}{\partial x_j}+\frac{\partial\langle v_j\rangle}{\partial x_i}\right)-\frac{\partial}{\partial x_j}(\langle p\rangle\,\delta_{ij}) \tag{A.7}$$

which can be written in the following form:

$$\frac{\partial\langle v_i\rangle}{\partial t}=\frac{\partial}{\partial x_j}\left[\frac{g(0)}{2}\varepsilon_{ijk}\langle v_k\rangle+S_{ij}\right] \tag{A.8}$$

where S_{ij} is symmetric. The rate of change of the mean impulse is thus

$$\frac{\partial\langle M_i\rangle}{\partial t}=\frac{2\rho g(0)}{3}\int\langle v_i\rangle\, d^3\mathbf{r} \tag{A.9}$$

This is not necessarily zero, and hence we see that conservation of the impulse and, if the existence and finiteness of the integral in (A.2) are given, the angular momentum are not compatible with Eq. (1). This suggests that Eq. (1) is not valid for incompressible fluids.

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